Improvement on the Controlled Cholesky Factorization Preconditioner for linear system arising in interior-point method

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Outline

- Linear Programming Problem
- Preconditioning in Interior Point Methods
- Controled Cholesky Factorization - CCF
- Modified Cholesky Factorization
- Numerical Results

- ***Primal problem***

$$\begin{array}{llll} min & c^T x & & \\ s.a & Ax & = & b, \\ & x & \geq & 0. \end{array} \quad (1)$$

$A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

- ***Dual problem***

$$\begin{array}{llll} \max & b^T y & & \\ s.a & A^T y + z & = & c, \\ & z & \geq & 0. \end{array} \quad (2)$$

where $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$.

- Optimality conditions for primal and dual problems:

$$\begin{array}{lcl} Ax & = & b \\ A^T y + z & = & c \\ XZe & = & 0 \\ x, z & \geq & 0 \end{array} \qquad (3)$$

where $X = diag(x)$, $Z = diag(z)$ and $e = (1, 1, ..., 1)^T$.

- Modified KKT conditions:

$$\begin{array}{lcl} Ax & = & b \\ A^T y + z & = & c \\ XZe & = & \mu e \\ x, z & \geq & 0 \end{array} \qquad (4)$$

where $\mu > 0$.

- The **search directions** is obtained by solving two linear systems, like:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} r_p \\ r_d \\ r_a \end{bmatrix} \tag{5}$$

where $r_p = b - Ax$, $r_d = c - A^T y - z$ e $r_a = -XZ$.

- In practice, the variables Δz and Δx can be eliminated and the systems reduces to :

$$A\Theta A^T \Delta y = r_p + A\left(\Theta r_d - Z^{-1} r_m\right). \tag{6}$$

where $\Theta = Z^{-1}X$.

- To solve such linear system is the **most expensive step** in an interior point method.

- The Normal Equations system

$$A\Theta A^T \Delta y = r_p + A\left(\Theta r_d - Z^{-1} r_m\right)$$

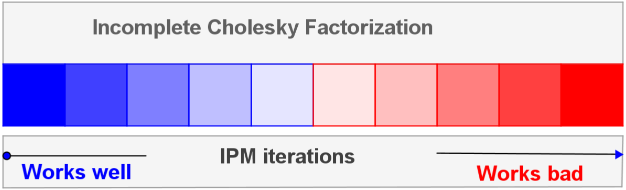
can be solved by **Preconditioned Conjugate Gradient Method.**

- Since the matrix Θ **become ill conditioned** along the IPM iterations, the matrix $A\Theta A^T$ become ill conditioned too.

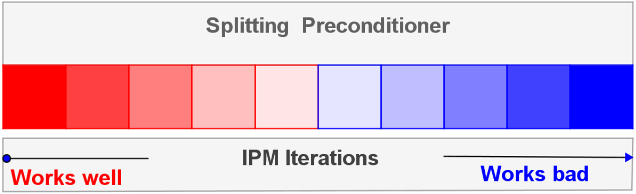
- **Good preconditioners** are necessary.

Preconditioning in Interior Point Method

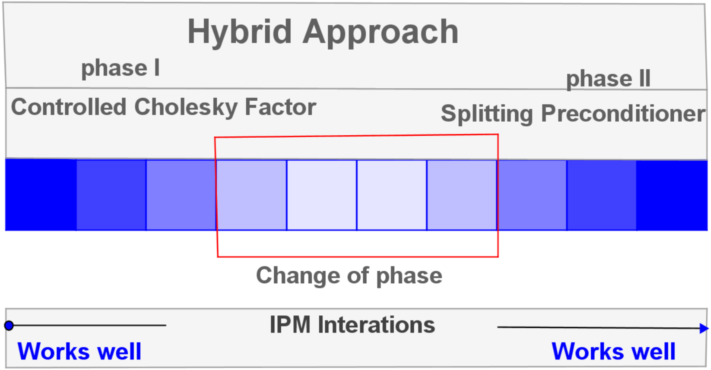
- **Incomplete Cholesky Factorization / Controlled Cholesky Factorization - CCF** (*F. F. Campos*):



- **Splitting Preconditioner** (*Oliveira and Sorensen*):



- **Hybrid Approach** (*Bocanegra, Campos e Oliveira*): Controlled Cholesky Factorization + Splitting.



- **Change of phase** is a critical moment and the **CCF plays a important role** in it.

- Consider the Cholesky factorization

$$A\Theta A^T = \underbrace{LL^T}_{\text{complete}} = \underbrace{\tilde{L}\tilde{L}^T}_{\text{incomplete}} + \underbrace{R}_{\text{remainder matrix}}$$

and set $E = L - \tilde{L}$;

- CCF is based in the minimization problem: $min||E||_F^2$=

$$min \sum_{j=1}^{n} \sum_{i=1}^{n} |l_{ij} - \tilde{l}_{ij}|^2 = min \sum_{j=1}^{n} \left[\sum_{k=1}^{m_j+\eta} |l_{i_k j} - \tilde{l}_{i_k j}|^2 + \sum_{k=m_j+\eta+1}^{n} |l_{i_k j}|^2 \right].$$

- It because when $\tilde{L} \simeq L$, then $E \simeq 0$ and

$$\tilde{L}^{-1}(A\Theta A^T)\tilde{L}^{-T} \simeq I.$$

Controlled Cholesky Factorization - CCF

Key features

- The preconditioner is calculated by incomplete factorization $A\Theta A^T = LDL^T$ where $D = diag(d_1, ..., d_m)$ is a diagonal matrix with entries $d_j > 0$;
- Choice entries by value (only the $\eta + m_j$ largest entries nonzero in absolute value are considered);
- Versatile Preconditioner

Table : Fill-in and drop-out with CCF(η)

η	**M**	**storage**
$-m$	$diag(A\Theta A^T)^{-1/2}$	less than $A\Theta A^T$
0	$\tilde{L}$	equal to $A\Theta A^T$
m	L	more than $A\Theta A^T$

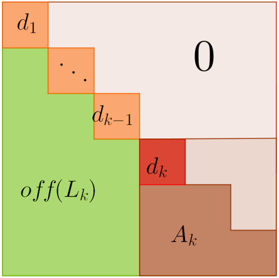
- Predictable storage

Deficient pivots and breakdowns in CCF

During the computation of the kth column of $\tilde{L}$:

If $d_k \simeq 0$;

- Discard the columns of $\tilde{L}$;
- Compute a shift $\alpha_i = 5,0 \times 10^{-4} \times 2^i$;
- Restart the CCF with $A\Theta A^T + \alpha_i I$;
- This process is repeated untill the constrution of $\tilde{L}$.



This process increases the preconditioning time.

Examples from Interior Point Methods

- **Breakdowns** in CCF cause the restarting of Cholesky Factorization several times in the same iteration of IPM.

	CCF restart number - IPM iterations									
IPM Iteration:	6	7	8	9	10	11	12	13	14	15
els19				14	14	13	13	12	12	*
ch25a						13	13	11	11	11
scr15		13	13	12	12	12	*			
rou20		15	14	13	13	13	13	*		

- CCF is restarted 81 times during the solution of the problem **rou20**.

Previously in CCF (Updated CCF)

1. Given an incomplete factorization $M = LDL^T$ of $A\Theta A^T$, a preconditioner for $A\Theta A^T + \alpha I$ is defined as

$$\tilde{M} = \tilde{L}\tilde{D}\tilde{L}^T$$

where

$$\tilde{d}_j = d_j + \alpha$$

and

$$\tilde{l}_{ij} = l_{ij}\frac{d_j}{d_j + \alpha},$$

for $j = 1, ..., n$ and $i = j + 1, ..., n$.

2. Important theoretical results about updated preconditioners was presented by **S. Bellavia et al, (2012).**

Fixed a number J. If during the factorization of the matrix $A\Theta A^T$ occur a breakdown, its diagonal is increased by shift $\alpha > 0$:

1. If the breakdowns occur in j-column and if
$$j \leq J,$$
then the factorizations can be restarted;
2. If the breakdowns occur in j-column with
$$j > J,$$
then the factorization can not be restarted and the previous columns are updated.

Numerical Experiments

- We set $J \simeq \frac{N}{3}$, where N is the number of column of matrix $A\Theta A^T$.
- Updates on CCF like was proposed by *L.M. Silva and Oliveira*;
- The Exponential shift $\alpha = 5,0 \times 10^{-4} \times 2^i$;
- Change of phases based on the heuristic proposed by *Velazco, Oliveira and Campos*;
- Experiments were performed on an intel Corei5, 8 GB RAM and 3.20 GHz. System Linux. **PCx code.**

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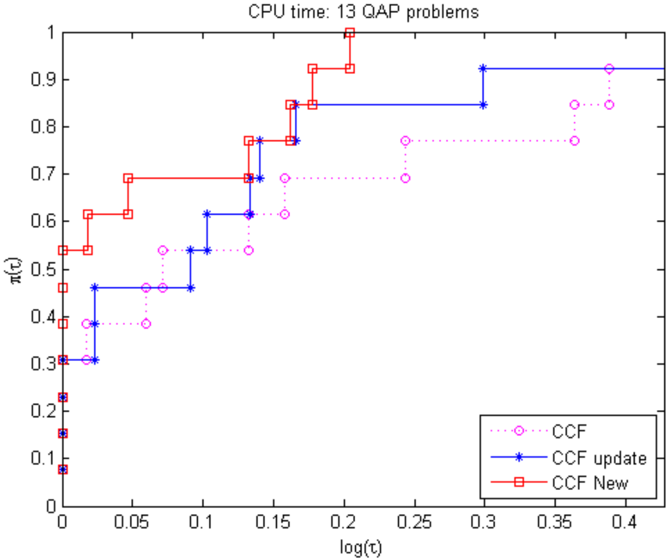
Problems Set and Statistics

Table : QAPLIB

Problem	Rows	Columns
els19	4350	13186
chr22b	5587	10417
chr25a	8149	15325
scr15	2234	6210
scr20	5079	15980
rou20	7359	37640
nug06	280	486
nug08	742	1632
nug12	3192	8856
nug15	6330	22275
qap08	742	1632
qap12	2794	8856
qap15	5698	22275

Computational Results - Performance Profile



Conclusions

- Updated CCF preconditioner was employed on a hybrid preconditioning approach in IPM context.
- Preliminary results show that the updated preconditioners can be improve the hybrid approach of preconditioning for IPM.
- Better results came from QAP problems.
- Other approaches for updating of preconditioner must be considered.
- others way of decisions, if the factorization can be restarted or not will be tested.

Thank you for your attention!

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Computational Results

Table : IPM Iterations

Problem	CCF	$CCF_{updated}$	CCF_{new}
els19	31	31	31
chr22b	29	29	29
chr25a	28	28	**27**
scr15	24	24	24
scr20	22	**20**	21
rou20	24	24	24
nug06	06	06	06
nug08	9	9	9
nug12	20	20	20
nug15	23	23	23
qap08	10	10	10
qap12	20	20	20
qap15	x	x	23

Computational Results

Table : CPU times

Problem	CCF	$CCF_{updated}$	CCF_{new}
els19	51,46	54,62	**50,86**
chr22b	20,79	20,43	**18,63**
chr25a	40,12	37,19	**36,61**
scr15	7,38	7,54	**7,08**
scr20	81,36	**63,24**	72,83
rou20	1160,16	994,28	**886,52**
nug06	0,08	0,08	0,08
nug08	**0,61**	0,62	0,63
nug12	**92,97**	114,37	105,14
nug15	**1325,51**	1460,64	1453,23
qap08	0,83	**0,79**	0,80
qap12	97,48	**82,34**	92,14
qap15	x	x	**1235,16**

Computational Results - Performance Profile

CPU time

IPM Iterations